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2e BOERHAAVESTRAAT 49
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AFDELING TOEGEPASTE WISKUNDE

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The deformation of a square wave by a linear filter

by

H.A. Lauwerier

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1. Introduction

We consider a linear filter L which has the property that it transforms an input function F(t) into the output function G(t) in a linear way. This means that the behaviour of L is sufficiently described by its response X(t) to a delta function input $\delta(t)$. Since the filter works upon the past only we have of course $X(t) = 0$ for $t < 0$. Since an arbitrary input function F(t) can be decomposed into delta functions by means of

(1.1) $$F(t) = \int_{-\infty}^{\infty} F(\tau)\ \delta\ (t-\tau)d\tau \ ,$$

we can find its response G(t) by means of linear compositions as

(1.2) $$G(t) = \int_{-\infty}^{t} F(\tau)\ X\ (t-\tau)d\tau \ .$$

As an application of this we have the following formulae for the response to a unit step function $\theta(t)$ and a harmonic input starting at $t = 0$ viz. $e^{i\omega t}\theta(t)$

(1.3) $$\delta(t) \doteqdot X(t)$$

(1.4) $$\theta(t) \doteqdot \int_{0}^{t} X(\tau)d\tau$$

(1.5) $$e^{i\omega t}\theta(t) \doteqdot e^{i\omega t} \int_{0}^{t} e^{-i\omega t}X(\tau)d\tau \ .$$

The latter formula shows that the response to the harmonic input tends asymptotically to a similar harmonic output but with a different complex factor

(1.6) $$L(i\omega) = \int_{0}^{\infty} e^{-i\omega t}X(t)dt$$

by which amplitude and phase are changed.
This suggests the use of Laplace transformation methods. This means formally that $i\omega$ is replaced by the complex variable p. We shall write

(1.7) $$f(p) = \int_{-\infty}^{\infty} e^{-pt} F(t)dt \; , \; g(p) = \int_{-\infty}^{\infty} e^{-pt} G(t)dt$$

and

(1.8) $$L(p) = \int_{-\infty}^{\infty} e^{-pt} X(t)dt \; .$$

Usually $f(p)$ and $g(p)$ exist at least in a vertical strip in the complex p-plane , $\alpha < \text{Re } p < \beta$, whereas $L(p)$ exists at least in a half-plane $\text{Re } p > \gamma$ owing to the fact that $X(t)$ vanishes for negative t. Assuming the existence of these transforms in a common strip the convolution relation (1.2) can be transformed into

(1.9) $$g(p) = f(p) \, L(p) \; .$$

This simple relation represents the source for the investigation of the properties of $G(t)$ by means of functiontheoretical tools.

As a first application we consider the following boundary value problem

(1.10) $$D \frac{\partial^2 c}{\partial x^2} = \frac{\partial c}{\partial t} \qquad \text{for } 0 < x < a \, , \, t > 0.$$

with

$$c = 0 \qquad \text{for} \quad t = 0 \, ,$$
$$c = 0 \qquad \text{for} \quad x = a \, ,$$
$$-\frac{\partial c}{\partial x} = F(t) \qquad \text{for} \quad x = 0 \; .$$

This problem may be translated in terms of conduction of heat although other interpretations are equally possible. Thus $c(x,t)$ may represent the temperature in a bar with one end kept at a fixed temperature and with an arbitrary input of heat at the other hand. Applying Laplace transformation we arrive at the following formula for the Laplace transform $\bar{c}$ of c

(1.11) $$\bar{c}(x,p) = f(p) \frac{\sinh q \, (a-x)}{q \cosh qa} \, ,$$

where $q = (p/D)^{\frac{1}{2}}$.

This is of the form (1.9). Thus the bar acts as a linear filter with an input function F(t) and as an output function the temperature at a given point x. Taking in particular the temperature at x = 0 the filter function L(p) appears as

$$(1.12) \qquad L(p) = \frac{\tanh qa}{q} .$$

For a harmonic input function $a^{-1} e^{i\omega t}\ \theta(t)$ the output is determined by

$$(1.13) \qquad g(p) = \frac{1}{p-i\omega}\ \frac{\tanh qa}{qa} .$$

The output function G(t) can then be found by using the complex inversion formula

$$(1.14) \qquad G(t) = \frac{1}{2\pi i} \int_L e^{pt} g(p)dp ,$$

where L is a vertical path in the domain of regularity of p. In this special case L may be any vertical in the half-plane Re p > 0. From (1.13) it follows that g(p) has simple poles at $p = i\omega$ and at $p = -(n+\frac{1}{2})^2\ \pi^2\ D/a^2$ for n = 0,1,2,... . The asymptotic behaviour of G(t) which in general is determined by the singularities with the largest real part is given here by the contribution from the pole at $p = i\omega$ only. This gives

$$(1.15) \qquad G(t) \approx \frac{\tanh a\sqrt{\frac{i\omega}{D}}}{a\sqrt{\frac{i\omega}{D}}}\ e^{i\omega t} .$$

Another interpretation of this result is that this gives the stationary response to a continuous harmonic input i.e. one which starts oscillating at minus infinity.

2. Square wave input

A square wave starting at t = 0 may be described by

$$(2.1) \qquad k(t) = \begin{cases} 1 & \text{for} \quad 2n < t < 2n+1 , \\ -1 & \text{for} \quad 2n+1 < t < 2n+2 , \end{cases}$$

where n = 0,1,2,... .

Another representation is

(2.2) $$k(t) = \theta(t)-2\theta(t-1)+2\theta(t-2)-2\theta(t-3)+\dots .$$

The Laplace transform becomes

(2.3) $$\mathcal{L}\ k(t) = p^{-1}(1-2e^{-p}+2e^{-2p}-2e^{-3p}+\dots) = p^{-1}\tanh\tfrac{1}{2}p .$$

Slightly more generally we have

(2.4) $$\mathcal{L}\ k(\omega t) = \frac{1}{p}\tanh\frac{p}{2\omega} .$$

Later on we shall need the original of

(2.5) $$\mathcal{L}\ l(t) = \frac{1}{p+\lambda}\ \tanh\frac{p}{2\omega} \quad \text{with} \quad \text{Re}\ \lambda > 0.$$

Applying the expansion (2.3) we find at once

(2.6) $$l(t) = e^{-\lambda t}\theta(t) - 2e^{-\lambda(t-\omega^{-1})}\theta(t-\omega^{-1})+2e^{-\lambda(t-2\omega^{-1})}\theta(t-2\omega^{-1})-\dots ,$$

which no longer represents a periodic wave form but which still has a periodic wave as its limit.

In fact in the time interval $2n < \omega t < 2n+1$ we have

$$l(t) = e^{-\lambda t}\ (1-2e^{\lambda/\omega}+2e^{2\lambda/\omega}-\dots+2e^{2n\lambda/\omega}) =$$

$$= e^{-\lambda t}\left\{\frac{2e^{(2n+1)\lambda/\omega}}{e^{\lambda/\omega}+1} - \frac{e^{\lambda/\omega}-1}{e^{\lambda/\omega}+1}\right\} .$$

As $n \to \infty$ we obtain as the asymptotic wave form

(2.7) $$l(t) \to \frac{2e^{-\lambda\tau/\omega}}{1+e^{-\lambda/\omega}} , \quad \omega t = 2n+\tau ,\ 0 < \tau < 1,$$

and similarly

(2.8) $$l(t) \to -\frac{2e^{-\lambda\tau/\omega}}{1+e^{-\lambda\omega}} ,\ \omega t = 2n+1+\tau ,\ 0 < \tau < 1.$$

The response of an arbitrary filter to a square wave input $k(\omega t)$ is determined by

(2.9) $$g(p) = \frac{1}{p}\tanh\frac{p}{2\omega}\,.\,L(p).$$

The usual procedure of obtaining the asymptotic wave form of the output G(t) consists in the expansion of the right-hand side of (2.9) with respect to the poles of the first factor. This amounts to the expansion of k(ωt) in a Fourier series

(2.10) $$k(\omega t) = \frac{4}{\pi\omega}\sum_{n=0}^{\infty}\frac{\sin(2n+1)\pi\omega t}{2n+1}.$$

The final result would be of the form

(2.11) $$G(t) \approx \frac{4}{\pi\omega i}\sum_{n=-\infty}^{\infty}\frac{1}{2n+1}L\{(2n+1)\pi\omega i\}\exp\{(2n+1)\pi\omega t i\}.$$

However, the expansion (2.10) converges so slowly that from a numerical point of view it is completely worthless. The expansion (2.11) is in general not much better. Moreover in both expansions the discontinuous character of the square wave form is completely masked.

The object of this note is to indicate another treatment by means of which the asymptotic wave form of the output can be determined much more precisely. The essential point consists in an appropriate expansion of the second factor of (2.9). At first we should remove the shift part of the operator L(p). This means that we shall write

(2.12) $$L(p) = e^{-\sigma p}L_1(p)$$

so that now $L_1(p)$ is bounded as Re $p \to -\infty$. Next $p^{-1}L_1(p)$ is expanded into a Mittag-Leffler series with respect to its poles. Thus we write

(2.13) $$\frac{1}{p}L(p) = e^{-\sigma p}\sum_{\lambda}\frac{c(\lambda)}{p+\lambda}.$$

If L(p) has one or more branchpoints this expansion may be generalized by the inclusion of continuous sums of integrals.
Substitution of (2.13) into (2.9) gives

(2.14) $$g(p) = e^{-\sigma p}\sum_{\lambda}c(\lambda)\frac{\tanh\frac{p}{2\omega}}{p+\lambda}.$$

Inversion is now simple matter in view of the results (2.5) sqq. For the asymptotic wave form in the "positive" interval we obtain at once

$$(2.15)\qquad G(t) \approx \sum_{\lambda} c(\lambda)\, \frac{2e^{-\lambda\tau/\omega}}{1+e^{-\lambda/\omega}} \qquad \text{as} \quad n \to \infty ,$$

where $\omega(t-\sigma) = 2n+\tau$, $0 < \tau < 1$.

This technique will be illustrated by considering the boundary value problem (1.10) but now for a square wave input $F(t) = a^{-1}k(\omega t)$.
The Laplace transform of the output function $G(t) = c(0,t)$ is now given by

$$(2.16)\qquad g(p) = \frac{\tanh qa}{qa}\, \frac{\tanh \frac{p}{2\omega}}{p} .$$

According to (2.13) with $\sigma = 0$ we may use the expansion

$$(2.17)\qquad \frac{\tanh qa}{pqa} = \frac{1}{p} + \sum_{n=0}^{\infty} \frac{c_n}{p+(n+\frac{1}{2})^2\pi^2 D/a^2} .$$

A simple calculation shows that

$$(2.18)\qquad c_n = -\frac{2}{\pi^2(n+\frac{1}{2})^2} .$$

Then (2.15) gives finally

$$(2.19)\qquad G(t) \approx 1 - \frac{4}{\pi^2} \sum_{n=0}^{\infty} \frac{1}{(n+\frac{1}{2})^2}\, \frac{e^{-\mu_n \tau}}{1+e^{-\mu_n}} ,$$

with

$$\mu_n = \frac{\pi^2 D}{\omega a^2} (n+\tfrac{1}{2})^2 .$$

The series of the right-hand side of (2.19) is very well suited for numerical calculations since all terms are positive and contain an exponential term. $G(t)$ can easily be plotted since it depends only on the dimensionless group

$$(2.20)\qquad \varepsilon = \frac{\pi^2 D}{\omega a^2} .$$

For a few values of ε the graph of $G(t,\varepsilon)$ is given below. We note that $\sum_{n=0}^{\infty} (n+\frac{1}{2})^{-2} = \frac{1}{2}\pi^2$ so that the following relations can be obtained at once

$$(2.21) \qquad \begin{cases} G(t,0) = 0 \quad , \quad G(t,\infty) = 1 \ , \\ G(0,\varepsilon) + G(1,\varepsilon) = 0 \ . \end{cases}$$